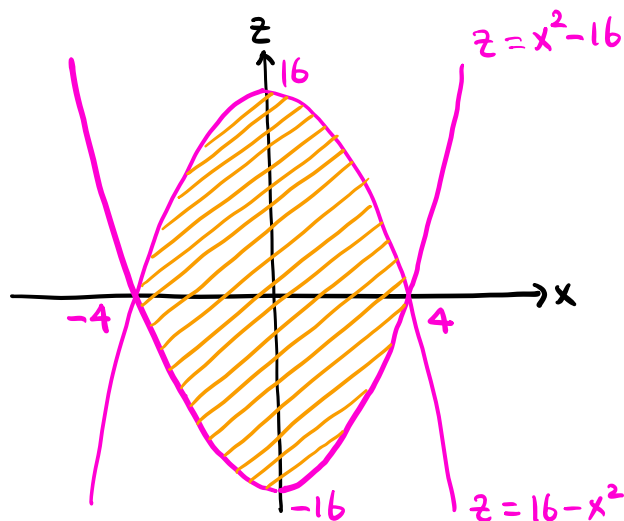


1. This problem has three parts.

- (a) (2 points) Sketch the curves $z = 16 - x^2$ and $z = x^2 - 16$ in the x - z plane. Determine the points of intersection of the two curves.



Intersections :

$$z = x^2 - 16 \text{ and } z = 16 - x^2$$

$$\Rightarrow x^2 - 16 = 16 - x^2 \Rightarrow x^2 = 16$$

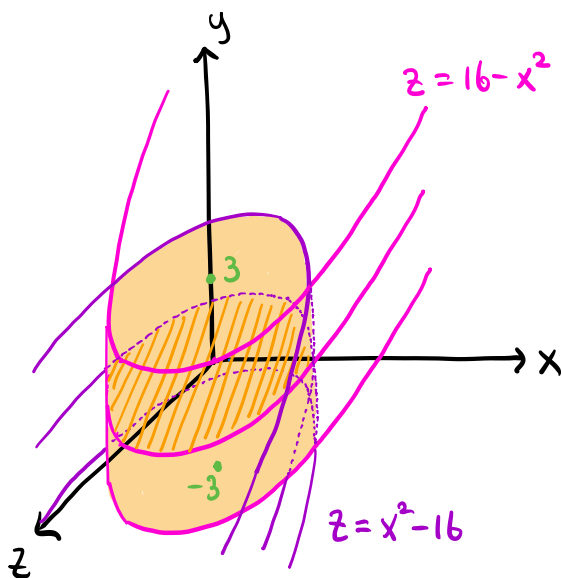
$$\Rightarrow x = \pm 4, z = 0.$$

- (b) (5 points) Find the area bounded by the curves $z = 16 - x^2$ and $z = x^2 - 16$ in the x - z plane.

The region is given by $-4 \leq x \leq 4$, $x^2 - 16 \leq z \leq 16 - x^2$

$$\text{Area} = \int_{-4}^4 \int_{x^2-16}^{16-x^2} 1 \, dz \, dx = \int_{-4}^4 (32 - 2x^2) \, dx = 32x - \frac{2}{3}x^3 \Big|_{x=-4}^{x=4} = \boxed{\frac{512}{3}}$$

- (c) (3 points) Find the volume bounded by the surfaces $z = 16 - x^2$, $z = x^2 - 16$, $y = 3$, and $y = -3$.



The solid is a cylinder with

height 6 and base area $\frac{512}{3}$

$$\Rightarrow \text{Volume} = \frac{512}{3} \cdot 6 = \boxed{1024}$$

Note You can also compute the volume by a triple integral.

$$\text{Volume} = \int_{-3}^3 \int_{-4}^4 \int_{x^2-16}^{16-x^2} 1 \, dz \, dx \, dy$$

2. Let $f(x, y, z) = \frac{x^3 + y^3 + z^3}{3} - xyz$.

(a) (4 points) Find ∇f and evaluate it at $(x, y, z) = (1, 2, 3)$.

$$\nabla f = (f_x, f_y, f_z) = (x^2 - yz, y^2 - xz, z^2 - xy)$$

$$\nabla f(1, 2, 3) = (-5, 1, 7)$$

(b) (3 points) Find the unit vector \mathbf{u} along which the directional derivative $D_{\mathbf{u}}f$ is maximum at $(x, y, z) = (1, 2, 3)$.

The direction of the maximum directional derivative is given by $\nabla f(1, 2, 3)$

$$\Rightarrow \vec{u} = \frac{\nabla f(1, 2, 3)}{|\nabla f(1, 2, 3)|} = \frac{(-5, 1, 7)}{\sqrt{(-5)^2 + 1^2 + 7^2}} = \frac{1}{\sqrt{75}}(-5, 1, 7)$$

(c) (3 points) Find a unit vector \mathbf{u} such that the directional derivative $D_{\mathbf{u}}f$ is zero.

Set $\vec{u} = (a, b, c)$

$$D_{\vec{u}}f(1, 2, 3) = \nabla f(1, 2, 3) \cdot \vec{u} = (-5, 1, 7) \cdot (a, b, c) = -5a + b + 7c$$

$$D_{\vec{u}}f(1, 2, 3) = 0 \Rightarrow -5a + b + 7c = 0$$

Take \vec{u} to be the unit vector of $(1, -2, 1)$

$$\Rightarrow \vec{u} = \frac{(1, -2, 1)}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{1}{\sqrt{6}}(1, -2, 1)$$

Note You can take \vec{u} to be any unit vector with

$$-5a + b + 7c = 0$$

e.g. $\vec{u} = \frac{1}{\sqrt{26}}(1, 5, 0)$ or $\frac{1}{5\sqrt{2}}(0, -7, 1)$.

3. Evaluate the triple integral $\int \int \int_B xyz \, dV$ for the following two choices of B :

(a) (5 points) B is the cube $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

$$\begin{aligned}\iiint_B xyz \, dV &= \int_0^1 \int_0^1 \int_0^1 xyz \, dz dy dx = \int_0^1 \int_0^1 \frac{xyz^2}{2} \Big|_{z=0}^{z=1} dy dx \\ &= \int_0^1 \int_0^1 \frac{xy}{2} dy dx = \int_0^1 \frac{xy^2}{4} \Big|_{y=0}^{y=1} dx \\ &= \int_0^1 \frac{x}{4} dx = \frac{x^2}{8} \Big|_{x=0}^{x=1} = \boxed{\frac{1}{8}}\end{aligned}$$

Note You can also compute the integral as follows:

$$\begin{aligned}\iiint_B xyz \, dV &= \int_0^1 \int_0^1 \int_0^1 xyz \, dz dy dx = \left(\int_0^1 x dx \right) \left(\int_0^1 y dy \right) \left(\int_0^1 z dz \right) \\ &= \frac{x^2}{2} \Big|_{x=0}^{x=1} \cdot \frac{y^2}{2} \Big|_{y=0}^{y=1} \cdot \frac{z^2}{2} \Big|_{z=0}^{z=1} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}\end{aligned}$$

(b) (5 points) B is the region $0 \leq x \leq y \leq z \leq 1$.

The bounds are $0 \leq z \leq 1, 0 \leq y \leq z, 0 \leq x \leq y$.

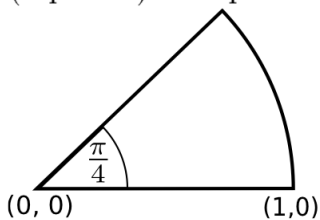
$$\begin{aligned}\iiint_B xyz \, dV &= \int_0^1 \int_0^z \int_0^y xyz \, dx dy dz = \int_0^1 \int_0^z \frac{x^2 y z}{2} \Big|_{x=0}^{x=y} dy dz \\ &= \int_0^1 \int_0^z \frac{y^3 z}{2} dy dz = \int_0^1 \frac{y^4 z}{8} \Big|_{y=0}^{y=z} dz \\ &= \int_0^1 \frac{z^5}{8} dz = \frac{z^6}{48} \Big|_{z=0}^{z=1} = \boxed{\frac{1}{48}}\end{aligned}$$

Note You can also write the integral as follows:

$$\iiint_B xyz \, dV = \int_0^1 \int_0^y \int_y^1 xyz \, dz dy dx$$

4. In both parts, the density is assumed to be $\rho(x, y) = x$.

(a) (5 points) The plot below

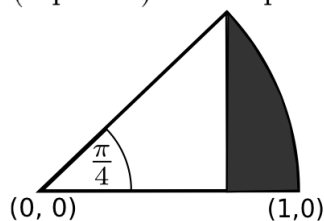


shows the sector of a circle of radius 1. The angle at the center is $\frac{\pi}{4}$ or 45° as indicated. Find the mass of the sector.

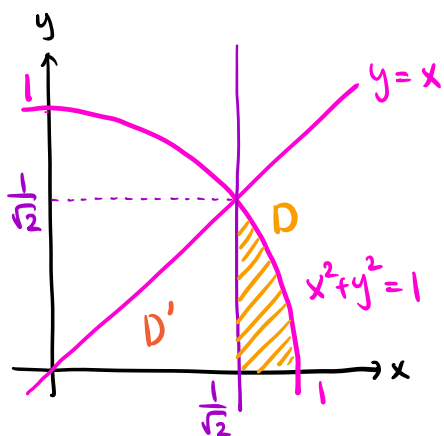
In polar coordinates, the domain D is given by $0 \leq \theta \leq \frac{\pi}{4}$, $0 \leq r \leq 1$.

$$\begin{aligned}
 m_D &= \iint_D \rho(x, y) \, dA = \iint_D x \, dA = \int_0^{\pi/4} \int_0^1 r \cos \theta \cdot r \, dr \, d\theta \\
 &= \int_0^{\pi/4} \frac{r^3}{3} \cos \theta \Big|_{r=0}^{r=1} \, d\theta = \int_0^{\pi/4} \frac{1}{3} \cos \theta \, d\theta = \frac{1}{3} \sin \theta \Big|_{\theta=0}^{\theta=\pi/4} = \boxed{\frac{\sqrt{2}}{6}}
 \end{aligned}$$

(b) (5 points) In the plot below



a part of the sector is shaded. Find the mass of the shaded region.



Intersection: $x^2 + y^2 = 1$ and $y = x$
 $\Rightarrow x^2 + x^2 = 1 \Rightarrow x = y = \frac{1}{\sqrt{2}}$

D : the shaded region

D' : the complement of D in the sector

D' is given by $0 \leq x \leq \frac{1}{\sqrt{2}}$, $0 \leq y \leq x$

$$\Rightarrow m_{D'} = \iint_{D'} \rho(x, y) \, dA = \int_0^{1/\sqrt{2}} \int_0^x x \, dy \, dx = \int_0^{1/\sqrt{2}} x^2 \, dx = \frac{x^3}{3} \Big|_{x=0}^{x=1/\sqrt{2}} = \frac{\sqrt{2}}{12}$$

$$m_D + m_{D'} = \frac{\sqrt{2}}{6} \Rightarrow m_D = \frac{\sqrt{2}}{6} - \frac{\sqrt{2}}{12} = \boxed{\frac{\sqrt{2}}{12}}$$

Note In polar coordinates, D is given by $0 \leq \theta \leq \frac{\pi}{4}$, $\frac{1}{\sqrt{2} \cos \theta} \leq r \leq 1$

5. In all the three parts, x and y are assumed to be positive with $xy = 4$.

- (a) (3 points) Find the point (x, y) , with $x, y > 0$, that is an extremum of $f(x, y) = \frac{x^2 + y^2}{2}$ subject to $g(x, y) = xy = 4$.

$$\text{Solve } \nabla f = \lambda \nabla g \text{ and } g = 4.$$

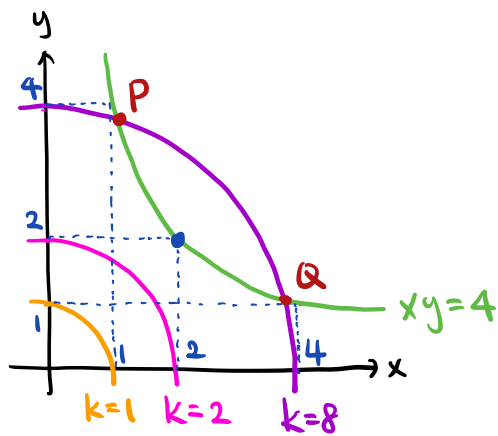
$$\Rightarrow (x, y) = \lambda (y, x) \text{ and } xy = 4$$

$$\Rightarrow \begin{cases} x = \lambda y \rightsquigarrow x^2 = \lambda xy \\ y = \lambda x \rightsquigarrow y^2 = \lambda xy \end{cases}$$

$$\Rightarrow x^2 = y^2 \Rightarrow x = y \Rightarrow (x, y) = \boxed{(2, 2)}$$

\uparrow $x, y > 0$ \uparrow $xy = 4$

- (b) (3 points) Sketch the curve $g(x, y) = xy = 4$ as well as the level curves of $f(x, y)$ with $f(x, y)$ being $\frac{1}{2}, 2, 8$, respectively, in the first quadrant of the x - y plane. Is the extremum of part (a) a minimum or a maximum? Explain.



$$\text{Level } k : f(x, y) = k \Rightarrow \frac{x^2 + y^2}{2} = k \Rightarrow x^2 + y^2 = 2k$$

$$\left. \begin{array}{l} k = \frac{1}{2} : x^2 + y^2 = 1 \\ k = 2 : x^2 + y^2 = 4 \\ k = 8 : x^2 + y^2 = 16 \end{array} \right\}$$

The curve $xy = 4$ intersects the level curve at 8.

$$f(2, 2) = 4 < 8.$$

$\Rightarrow (2, 2)$ cannot be a maximum on the curve $xy = 4$.

$\Rightarrow (2, 2)$ must be a minimum

(c) (4 points) Find the minimum value of $\frac{x^3}{3} + \frac{2y^{3/2}}{3}$ subject to $xy = 4$ and x, y positive.

$$\text{Set } f(x, y) = \frac{x^3}{3} + \frac{2y^{3/2}}{3} \text{ and } g(x, y) = xy - 4.$$

$$\text{Solve } \nabla f = \lambda \nabla g \text{ and } g = 0.$$

$$\Rightarrow (x^2, y^{1/2}) = \lambda(y, x) \text{ and } xy - 4 = 0$$

$$\Rightarrow \begin{cases} x^2 = \lambda y \rightsquigarrow x^3 = \lambda xy \\ y^{1/2} = \lambda x \rightsquigarrow y^{3/2} = \lambda xy \end{cases}$$

$$\Rightarrow x^3 = y^{3/2} \Rightarrow x = y^{1/2}$$

$$xy - 4 = 0 \Rightarrow y^{1/2} \cdot y - 4 = 0 \Rightarrow y^{3/2} = 4 \Rightarrow y = 4^{2/3}, x = 4^{1/3}.$$

$$\text{The minimum value is } f(4^{1/3}, 4^{2/3}) = \boxed{4}$$

Note You can also solve this by removing the constraint.

$$xy = 4 \Rightarrow y = \frac{4}{x}$$

$$\frac{x^3}{3} + \frac{2y^{3/2}}{3} = \frac{x^3}{3} + \frac{2}{3} \left(\frac{4}{x}\right)^{3/2} = \frac{x^3}{3} + \frac{16}{3x^{3/2}} = \frac{x^3}{3} + \frac{16}{3} x^{-3/2}$$

The minimum occurs at the critical point.

$$\frac{d}{dx} \left(\frac{x^3}{3} + \frac{16}{3} x^{-3/2} \right) = x^2 - 8x^{-5/2}$$

$$x^2 - 8x^{-5/2} = 0 \Rightarrow x^{9/2} = 8 \Rightarrow x = 4^{1/3} \Rightarrow \frac{x^3}{3} + \frac{16}{3} x^{-3/2} = 4$$

The minimum value is 4.

6. Consider the surface given by $x^2 + y^2 + 3z^2 = 3$ and the plane $x + y + z = 1$.

(a) (3 points) Explain why the plane intersects the surface.

The point $(0, 0, 1)$ lies on the surface $x^2 + y^2 + 3z^2 = 3$ and the plane $x + y + z = 1$.

\Rightarrow The surface and the plane intersect.

(b) (7 points) Suppose the curve along which the plane intersects the surface is C . Find the maximum and minimum values of z for points on C .

The constraints are $x^2 + y^2 + 3z^2 = 3$ and $x + y + z = 1$.

$$x + y + z = 1 \rightsquigarrow x = 1 - y - z.$$

$$x^2 + y^2 + 3z^2 = 3 \rightsquigarrow (1 - y - z)^2 + y^2 + 3z^2 = 3.$$

We find the extreme values of z subject to the constraint $(1 - y - z)^2 + y^2 + 3z^2 = 3$.

$$\text{Set } f(y, z) = z \text{ and } g(y, z) = (1 - y - z)^2 + y^2 + 3z^2 - 3.$$

$$\nabla f = \lambda \nabla g \Rightarrow (0, 1) = \lambda(-2(1 - y - z) + 2y, -2(1 - y - z) + 6z)$$

$$\Rightarrow 0 = \lambda(4y + 2z - 2), \quad 1 = \lambda(2y + 8z - 2).$$

$\lambda \neq 0$ by the second equation

$$\Rightarrow 0 = 4y + 2z - 2 \Rightarrow z = 1 - 2y.$$

$$(1 - y - z)^2 + y^2 + 3z^2 = 3 \Rightarrow (1 - y - 1 + 2y)^2 + y^2 + 3(1 - 2y)^2 = 3$$

$$\Rightarrow 14y^2 - 12y = 0 \Rightarrow y = 0, \frac{6}{7}$$

$$\Rightarrow (y, z) = (0, 1), \left(\frac{6}{7}, -\frac{5}{7}\right)$$

$$\Rightarrow \text{Maximum} = 1, \text{ minimum} = -\frac{5}{7}$$

7. The surface $x^2 + y^2 + z^2 = 1$ is the sphere, and as you know, the area of the upper hemisphere with $z > 0$ is 2π . In this problem, you will find areas of portions of that surface and your answers should therefore be less than 2π .

(a) (4 points) Find the area of the upper hemisphere inside the cylindrical surface $x^2 + y^2 = \frac{1}{4}$.

The domain D is given by $x^2 + y^2 \leq \frac{1}{4}$.

In polar coordinates, D is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \frac{1}{2}$.

$$x^2 + y^2 + z^2 = 1 \text{ and } z > 0 \Rightarrow z = \sqrt{1 - x^2 - y^2}$$

$$\text{Area} = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1-x^2-y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{1-x^2-y^2}}$$

$$\Rightarrow \text{Area} = \iint_D \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}} dA = \iint_D \frac{1}{\sqrt{1-x^2-y^2}} dA$$

$$= \int_0^{2\pi} \int_0^{1/2} \frac{r}{\sqrt{1-r^2}} dr d\theta = \int_0^{2\pi} \int_1^{3/4} u^{-1/2} \cdot \left(-\frac{1}{2}\right) du d\theta$$

← Jacobian
↑
 $u = 1 - r^2$

$$= \int_0^{2\pi} -u^{1/2} \Big|_{u=1}^{u=3/4} d\theta = \int_0^{2\pi} 1 - \frac{\sqrt{3}}{2} d\theta = \boxed{\pi(2 - \sqrt{3})}$$

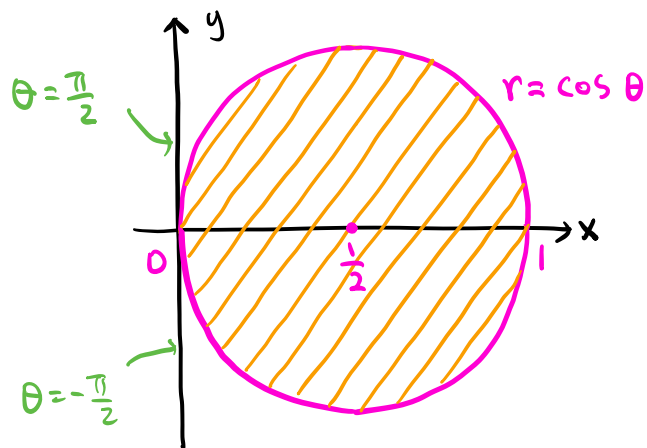
Note You can find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using implicit differentiation.

The sphere is a level surface of $f(x, y, z) = x^2 + y^2 + z^2$.

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{y}{z}$$

(b) (6 points) Find the area of the upper hemisphere inside the cylindrical surface $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$.

The domain D is given by $(x - \frac{1}{2})^2 + y^2 \leq \frac{1}{4}$.



In polar coordinates,

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4} \rightsquigarrow x^2 - x + y^2 = 0$$

$$\rightsquigarrow x = x^2 + y^2 \rightsquigarrow r \cos \theta = r^2$$

$$\rightsquigarrow r = \cos \theta$$

D is given in polar coordinates by $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq \cos \theta$.

$$\Rightarrow \text{Area} = \iint_D \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}} = \iint_D \frac{1}{\sqrt{1-x^2-y^2}} dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\cos \theta} \frac{r}{\sqrt{1-r^2}} dr d\theta = \int_{-\pi/2}^{\pi/2} \int_1^{\sin^2 \theta} u^{-1/2} (-\frac{1}{2}) du d\theta$$

← Jacobian

$$= \int_{-\pi/2}^{\pi/2} -u^{1/2} \Big|_{u=1}^{u=\sin^2 \theta} d\theta = \int_{-\pi/2}^{\pi/2} 1 - |\sin \theta| d\theta$$

$$= 2 \int_0^{\pi/2} 1 - \sin \theta d\theta = 2(\theta + \cos \theta) \Big|_{\theta=0}^{\theta=\pi/2} = \boxed{2(\frac{\pi}{2} - 1)}$$

$$= \int_{-\pi/2}^{\pi/2} \int_1^{\sin^2 \theta} u^{-1/2} \cdot \left(-\frac{1}{2}\right) du d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -u^{1/2} \Big|_{u=1}^{u=\sin^2 \theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 1 - |\sin \theta| d\theta$$

$$= 2 \int_0^{\pi/2} 1 - \sin \theta d\theta$$

$$= 2 (\theta + \cos \theta) \Big|_{\theta=0}^{\theta=\pi/2}$$

$$= \boxed{2 \left(\frac{\pi}{2} - 1 \right)}$$